ZDM Mathematics Education (2015) 47:377–390
DOI 10.1007/s11858-015-0679-2

ORIGINAL ARTICLE

Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: a case study

Joan Moss · Zachary Hawes · Sarah Naqvi · Beverly Caswell

Accepted: 27 January 2015 / Published online: 25 February 2015

Abstract Increased efforts are needed to meet the demand for high quality mathematics in early years classrooms. Despite the foundational role of geometry and spatial reasoning for later mathematics success, the strand receives inadequate instructional time and is limited to concepts of static geometry. Moreover, early years teachers typically lack both content knowledge and confidence in teaching geometry and spatial reasoning. We describe our attempt to deal with these issues through a research initiative known as the Math for Young Children project. The project integrates effective features of both design research and Japanese Lesson Study and is designed to support teachers in developing content knowledge and new approaches for teaching geometry and spatial reasoning. Central to our Professional Development model is the integration of four adaptations to the Japanese Lesson Study model: (1) teachers engaging in the mathematics, (2) teachers designing and conducting task-based clinical interviews, (3) teachers and researchers co-designing and carrying out exploratory lessons and activities, and (4) the creation of resources for other educators. We present our methods and the results of our adaptations through a case study of one Professional Learning Team. Our results suggest that the adaptations were effective in: (1) supporting teachers' content knowledge of and comfort level with geometry and spatial reasoning, (2) increasing teachers' perceptions of young children's mathematical competencies, (3) increasing teachers' awareness and commitment for the inclusion of high quality geometry and spatial reasoning as a critical component of early years mathematics, and (4) the creation of innovative resources for other educators. We conclude with theoretical considerations and implications of our results.

Keywords Teacher professional development · Geometry · Spatial reasoning · Early years mathematics · Lesson study

1 Background to our study

1.1 The necessity and challenges of providing high-quality mathematics for young children

There has been a growing call for the inclusion of more rigorous geometry and spatial reasoning in early years mathematics curricula (NCTM 2006). This comes as part of an unprecedented political as well as academic focus on the importance of mathematics in early years classrooms (Clarke et al. 2011; Ginsburg et al. 2008; MacDonald et al. 2012). The National Association for the Education of Young Children (NAEYC 2010) and the NCTM (2000) have jointly identified mathematics as a crucial area of learning for young children and have called for the provision of "high-quality, challenging, and accessible mathematics education for all 3- to 6-year-old children" (NAEYC 2010 p. 1).

Research indicates that young children have an intuitive and powerful grasp of "everyday mathematics" (Ginsburg et al. 2008 p. 3), a foundation upon which formal school mathematics is built. However, not all young children possess—nor have they been exposed to—the same quality of everyday mathematics. Indeed, by the time children enter formal schooling, there are striking differences in their readiness to engage in mathematical activity; differences often attributable to factors related to socioeconomic status (SES). These SES differences pose a major dilemma in

J. Moss (✉) · Z. Hawes · S. Naqvi · B. Caswell
Department of Applied Psychology and Human Development, Ontario Institute for Studies in Education, University of Toronto, Toronto, ON M5R 2X2, Canada
e-mail: joan.moss@utoronto.ca; jmoss@oise.utoronto.ca

early education contexts, especially with respect to mathematics and spatial reasoning where early competencies in these areas have been found to be strong predictors of later academic success (Verdine et al. 2013). For example, Duncan and colleagues (2007), in what has become a seminal study, showed that mathematics skills measured at kindergarten were strongly predictive of later academic success, above and beyond the variance accounted for by reading, attentional, and socioemotional skills.

An accumulating body of evidence points to the importance of spatial reasoning as a foundational skill that is not only related to later success in mathematics, but also in the arts and STEM disciplines of science, technology, and engineering (Wai et al. 2009).

Unfortunately, as is the case with mathematics, young children's spatial reasoning skills are not immune to the often detrimental effects of low SES (Verdine et al. 2013). Children as young as 3 years of age who come from low SES backgrounds have been found to demonstrate significantly lower spatial skills than their higher SES peers (Verdine et al. 2013). The evidence points to the fundamental importance of mathematics and spatial reasoning for later mathematical and academic success.

In the interest of equitable starting points for students of all backgrounds, the National Research Council (US) has urged policy makers to provide young children with "extensive, high-quality early mathematics instruction that can serve as a sound foundation for later learning in mathematics and contribute to addressing long-term systemic inequities in educational outcomes" (Cross et al. 2009 p. 2). To achieve this objective, however, requires dealing with a number of obstacles particularly apparent in the teaching and learning of geometry and spatial reasoning (Ginsburg et al. 2006).[1]

1.2 Obstacles preventing high-quality early years mathematics instruction

A major barrier to the implementation of a strong early years mathematics program lies in the inadequate preparation of early years educators. As many researchers have found, early years teachers are typically most comfortable teaching reading and other language-oriented skills and often lack confidence and interest in their own math knowledge (Copley 2004; Ginsburg et al. 2008; Hachey 2013). Given the mounting evidence linking teacher math content knowledge to student outcomes, this is clearly a serious issue (Hill et al. 2005).

Further, there is a widespread, and mistaken, assumption that young children are not interested in mathematics, nor are they capable of engaging in any kind of abstract mathematics. Consequently, many educators' view of early years mathematics is confined to rote approaches "counting, adding, subtracting, and knowing shapes" (Copley 2004 p. 405). In many jurisdictions, teachers and policy makers are concerned about the appropriateness of including *any* kind of formal math teaching in early years classrooms.

1.3 Geometry and spatial reasoning as a neglected priority in early years mathematics

In no area of early years mathematics is the lack of teacher preparation, lack of content knowledge, and lack of interest more evident than in the areas of geometry and spatial reasoning (Ginsburg et al. 2006). As Clements and Sarama (2011) have concluded, these areas of math are "often ignored or minimized in early education" (p. 133). The National Research Council (2006) points out that these areas have received significantly less research emphasis in the math education literature than has numeracy, despite the NCTM call for a significant time allotment to these topics in early years classrooms (NCTM 2006).

A recent Ontario survey of teachers in early years classrooms (K—2nd Grade) tells what seems to be a typical story. When asked to rank the amount of instructional time devoted to the five strands of the mathematics curriculum, kindergarten teachers ranked geometry and spatial sense as fourth out of the five strands. The first grade teachers reported that they devoted the least amount of time to the teaching of geometry and spatial sense and were least comfortable teaching these subjects (Bruce et al. 2012).

Reports from the United Kingdom (Jones 2000), Australia (MacDonald et al. 2012) and in the United States (Clements and Sarama 2011) further corroborate these findings and suggest that the lack of focus on early geometry learning is an international concern. For example, a recent US study tested 81 kindergarten teachers, more than half whom had MA degrees or higher, on their mathematical

[1] Throughout this paper we will be referring to geometry and spatial reasoning as both separate and unified subject areas. This decision is based on differences in the way spatial reasoning and geometry are conceived, studied, and discussed in the psychological versus mathematics education literature. The term spatial reasoning or spatial thinking will be used when referring to work conducted by psychologists or cognitive psychologists. Geometry and spatial reasoning will be used to reflect the work of mathematics educators and researchers. Whereas psychologists and cognitive scientists generally study spatial reasoning or spatial thinking as a collection of cognitive skills and processes, mathematics educators generally consider geometry and spatial reasoning as a unified strand of mathematics having to do more with geometrical concepts than spatial skills per se. Within our own work in mathematics education, we consider geometry and spatial thinking as closely linked, i.e., geometry as the study of spatial relationships. We also recognize that various spatial cognitive skills (e.g., visualization) are necessarily part of understanding certain geometric concepts (e.g., composing/decomposing 2D shapes). In this way, we see the importance of simultaneously developing children's geometric and spatial skills alongside conceptual understandings in order to support a deeper and more useful understanding of geometry and spatial reasoning.

content knowledge of number sense, patterning, ordering, shapes, spatial sense, and comparison problems. The teachers scored lowest in their knowledge of spatial sense, obtaining a mean score of 44.23 compared to a mean score of 89.12 on the test of number sense (Lee 2010). A search of the mathematics education literature in early years reveals that with the exception of a handful of programs (Casey et al. 2008; Clements and Sarama 2007; Levenson et al. 2011; Van den Heuvel-Panhuizen and Buijs 2005) there has been an absence of focus on both teacher PD and on young children's learning in the area of geometry and spatial reasoning.

The concern over the limited focus on geometry in the mathematics curriculum up through the grades has been a longstanding issue in the field of mathematics education. Freudenthal (1981) asserted that geometry is the most neglected subject of mathematics teaching and referred to geometry as the "forgotten" strand in pre-K through grade 12 mathematics curricula. Van den Heuvel-Panhuizen (2008) urges us to provide "more space for geometry in primary school" (p. 145).

The lack of attention paid to geometry and spatial thinking is concerning given its fundamental importance to the development of mathematics. As Clements and Sarama (2011) point out, geometry is a special kind of language through which we communicate ideas that are essentially spatial, "from number lines to arrays, even quantitative, numerical, and arithmetical ideas rest on a geometric base" (p. 134). In addition, they note that geometry spans mathematics and science and is central to other disciplines such as physics, chemistry, biology, geology, geography, art, and architecture. Furthermore, the work of developmental psychologists solidifies the importance of spatial thinking in mathematics in general. Over a century of psychological research supports the close relationship between spatial and mathematical processes, so much so that Mix and Cheng (2012) claim that "the relation between spatial ability and mathematics is so well established that it no longer makes sense to ask whether they are connected" (p. 206). A recent meta-analysis has shown that spatial reasoning is malleable and can be improved with practice and training in people of all ages (Uttal et al. 2013). Furthermore, Newcombe et al. (2013) suggest that by providing instructional support to enhance young children's spatial reasoning, we are opening up possibilities for their long-term interest and engagement with STEM-related activities. More specifically, Verdine et al. (2014), suggest that incorporating spatial reasoning into early years math, will have a "two-for-one" effect that yields benefits both for spatial reasoning as well as mathematical development (p. 6).

Indeed, the time is right for researchers and educators to implement and develop engaging geometry and spatial reasoning curricula that go beyond labeling and classifying shapes. As Clements and Sarama (2011) suggest, more research and creative efforts are needed in order to improve the practice of teaching in these areas. A primary means to achieve this objective is through the implementation of PD models that specifically address the aforementioned obstacles and promote the teaching and learning of early years geometry and spatial reasoning.

2 The Math for Young Children Project: a professional development model

2.1 History and general overview of the Math for Young Children Project

Since 2011, we have been working on a design research PD project for geometry and spatial reasoning in early years classrooms. The Math for Young Children (M4YC) project involves teachers, early childhood educators (ECEs), and school administrators. Our goal has been to strengthen and broaden teachers' content knowledge of, and curricula for, geometry and spatial reasoning in the early years.

To date, the M4YC project has involved working with seven district school boards throughout Ontario. We have collaborated with over 15 teacher-researcher teams, varying in size from 7 to 25 members consisting of K-2 teachers and ECEs, school administrators, district and provincial mathematics facilitators, and university mathematics educators and researchers. Hereafter, we refer to these teams as Professional Learning Teams (PLTs). In total, we have worked with over 100 early years educators and by extension their students ($N = 2,250+$). Our work has typically been carried out in underserved populations in schools with low provincial standardized test scores.

2.2 General overview of research objectives of M4YC

Our goals as a design research team involve cycles of design and analysis (Cobb et al. 2015 in press) and are both pragmatic and theoretical. Pragmatically, our goals involve investigating and improving designs for supporting the learning of both teachers and children. With regard to teacher learning, our goals are twofold: first, to investigate approaches to extend and enhance PD practices to help early years teachers gain content knowledge of geometry; secondly, to broaden teachers' understandings of their students' capabilities and interests in geometry and spatial reasoning. With regard to student learning, our goal is to design and implement assessments and curricula that focus explicitly on the development of dynamic and spatial aspects of geometry. A natural byproduct of this goal is the creation of field-tested resources that become available for use by other educators.

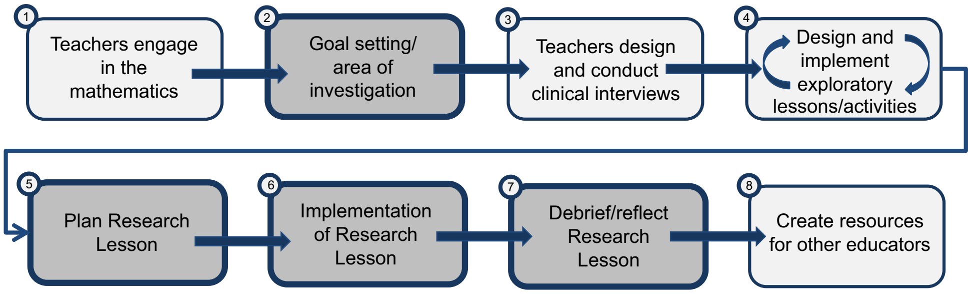

Fig. 1 Integrated Professional Development model: an adapted version of Japanese Lesson Study. The *darker shaded boxes* (stages 2, 5, 6, and 7) refer to the widely recognized four-stage model, while the *lighter shaded boxes* refer to the four additional adaptations

In the theoretical domain, we develop, test and revise conjectures regarding new approaches to the teaching and learning of geometry and spatial reasoning. Our project involves us in "engineering" our participants' development through new forms of practice, while, at the same time, systematically studying the effectiveness of those practices from the perspectives of both teacher change and student learning (Cobb et al. 2015 in press).

We recognize the need to reconceptualize what it means for teachers to learn and teach geometry in the early years. Rather than approaching geometry as a subject that is largely static in nature and one mainly concerned with labeling and classifying shapes (Clements 2004), we introduce teachers and administrators to the idea of geometry as dynamic, spatial, and imaginative in nature.

2.3 Methods to achieve our research objectives: adaptations of Japanese Lesson Study

To achieve our multifaceted research objectives we have adopted a Japanese Lesson Study (JLS) framework—a PD practice that has gained international interest because of its role in strong math performance in Japan (Huang et al. 2011; Lewis et al. 2006; White and Lim 2008). Japanese Lesson Study involves collaborative planning, teaching and reflecting on classroom lessons and is generally characterized by four steps: (1) goal setting/investigation; (2) planning; (3) implementation and Research Lesson; and (4) debriefing/reflection (e.g., Lewis et al. 2006). The JLS framework is highly collaborative and grounded in practice, two features of PD that have been identified as powerful in supporting change in both teachers and in the math performance of their students (Darling-Hammond et al. 2009; Garet et al. 2001). We conjectured that the collaborative nature of the lesson study process, and the opportunity to engage with these new approaches in their own classrooms, would support the teachers in both their content knowledge and confidence (Moss et al. 2012).

Our lesson study approach has evolved over time, and now includes four additional features, or adaptations, to the classic four-stage cycle. Thus, our adapted lesson study cycle is more closely aligned with an eight-stage cycle (see Fig. 1). The new adaptations include: (1) teachers engaging in the mathematics, (2) teachers designing and conducting task-based clinical interviews, (3) teachers and researchers co-designing and carrying out exploratory lessons and activities, and (4) the creation of resources for other educators. We conjectured that the inclusion of these four adaptations would strengthen the lesson study process and provide optimal support for teacher growth in both their content knowledge and their attitudes towards geometry and spatial reasoning.

To illuminate these adaptations, both their content and effectiveness, we present a case study of one PLT's learning experiences throughout a five-month PD cycle. We offer qualitative analyses of the effects these had on teacher learning. First, though, in this next section, we provide a brief overview of the four adaptations, our rationale for integrating them into the PD model, and our conjectures regarding their effects on teacher PD.

2.3.1 Adaptation #1: teachers engaging in the mathematics

In this adaptation, PLT's participated in a range of geometry and spatial reasoning activities as learners. This approach departs from the traditional lesson study process, whereby teachers generally begin by identifying mathematics topics that their students find challenging (such as subtraction with regrouping) to serve as the main focus for their inquiry. In our process, we invite the PLT's to work on math challenges not typically addressed in elementary geometry curricula such as mental transformations, spatial visualization, and the composition/decomposition of 3D shapes. We anticipated that teachers would become intrigued with these new types of math problems and that through participating in the various activities, the PLT members would see the benefit of trying similar activities in their own classrooms. We wondered if trying these unfamiliar tasks might begin to shift the PLT's attention away from the notion of geometry as static (e.g., naming and classifying shapes; teacher practices familiar to team members) towards a more dynamic and spatial view of geometry.

2.3.2 Adaptation #2: teachers designing and conducting clinical interviews

The second adaptation involves the practice of teachers and researchers co-designing and conducting clinical interviews. There are two main reasons for integrating this adaptation into our PD model. First, it was expected, and has been our experience to date, that clinical interviews naturally follow as a result of the first adaptation—the teachers trying novel math problems. We anticipated and repeatedly found that because the PLTs had worked on mathematics challenges not directly tied to the early years curriculum (e.g., mental transformations), they would begin to question how their students might fare on similar tasks. The second reason is based on growing evidence that shows the effectiveness of using clinical interviews to gain insight into children's thinking in a particular area of mathematics (Clarke et al. 2011; Ginsburg 1997).

First introduced by Jean Piaget, the clinical interview method has traditionally resided in the toolkit of psychologists and mathematics education researchers. The introduction and use of teacher-administered clinical interviews is a recent phenomenon in mathematics education and has been shown to be extremely effective in supporting teacher change. Perhaps the largest study involving teachers' use of clinical interviews has been reported by Clarke et al. (2011). The findings showed that clinical interviews enhanced teachers' mathematical content knowledge and pedagogical content knowledge (e.g., Ball et al. 2008), supported teachers' increased knowledge of students' reasoning strategies, and supported teacher confidence. Importantly, many teachers were surprised at how capable their students were. In the words of one of the teachers in the study: "my greatest surprise was that most children performed significantly better than I had anticipated" (Clarke 2011 p. 907). Taken together, our conjectures and the above findings provide a convincing case for including teacher-conducted clinical interviews as part of our PD.

2.3.3 Adaptation #3: exploratory lessons and activities

The next series of team meetings involve the design and implementation of what are referred to as "exploratory lessons" (Bruce and Ladky 2011). These lessons are co-designed by the group and taught to children as experiments in different settings. The focus for these lessons arises from speculations based on the PLT members' personal math explorations and analyses of the clinical interviews. In our work, we devote full days of the PD for such experimental lessons. In the early part of the meeting, group members work together to design tasks/mini lessons to address specific questions related to student reasoning. In the afternoon teachers select one or two students from their classrooms and invite them to the meeting room (often school library) to participate in the newly designed lesson, which is taught by a pair of teachers. This leaves the remaining members of the PLT to observe and to take notes. After the exploratory lesson, the group convenes to discuss observations and ways to improve the lesson. This newly revised version of the lesson is tried immediately with another small group of children. And finally, at the end of the meeting, each teacher makes a commitment to try the revised lesson in their own classroom and report in subsequent meetings, often through video footage, on how their students responded. In this way, exploratory lessons are both iterative and instructional in nature, helping the group move forward in their understanding of both pedagogy and student learning. This process allows us to focus on student reasoning, teacher "moves" and pedagogical approaches that directly influence student thinking.

We have a number of conjectures regarding the incorporation of exploratory lessons: firstly, having teachers involved in the design and analyses of lessons strengthens teachers' sense of efficacy and identity as co-researchers of the project. We also anticipate that this firsthand gathering and examination of the data involving the entire team gives each team member a better understanding and knowledge of developmental issues in the learning of various geometry strands. Finally, the exploratory lesson process where teachers watch each other teach is likely to contribute to new insights into their own teaching and a greater appreciation and awareness of the nuances of pedagogy.

2.3.4 Adaptation #4: creation of resources for spatial reasoning and geometry in early years

In one of the last steps in the process, the PLT engages in the creation of two types of resources. First, in anticipation of the final Public Research Lesson, the PLT prepares a presentation and lesson study package for guests who will attend the final Public Lesson day. The resources created for the public lesson day are designed to serve as background to the specific research lesson presented. Typically, the package includes an overview of the main math concepts in the lesson, objectives of the lesson, summaries of the relevant research literature, and the specific trajectories underpinning the design of the lesson. As is typical for Japanese Lesson study, the group also prepares an observation guide for guests to complete while observing the lesson.

The second set of resources—our fourth adaptation—is primarily web-based and created after the public lesson day. The creation of these resources involves the group in reviewing the full PD process and selecting activities, lessons, and clinical interviews they consider to be useful for other educators. These web-based resources typically include lessons, activities, games, video excerpts, student work samples, and, in some cases (as reported here), eBooks.

Initially, our purpose in including this fourth adaptation was to serve us, as researchers, with an authentic (and non-threatening) way to evaluate the impact of the PD project. In other words, the shared act of creating resources for others enables us as researchers to assess: *What aspects of the PD stood out most for the PLT? What aspects of our work together went unnoticed or received little attention?* Finally in terms of the group's learning: *Did the PD positively affect the group members' content knowledge of and attitudes towards the teaching and learning of geometry and spatial reasoning as a result of their participation in the M4YC process?*

In addition, as a research team we often had the benefit of being able to see our resources used in settings outside of our research. Thus, we were better able to test the validity of the resources and determine which lessons and activities proved to be most useful across a diversity of authentic classroom settings. This feedback has been extremely helpful in our subsequent work with other PLTs.

2.3.5 Summary of conjectures

In summary, it was conjectured that the four adaptations would support us in our goals to support change and growth in our PLT's. Specifically we anticipated that by inviting educators to experience and work on novel geometry challenges (Adaptation 1) we might begin the process of promoting a broadened vision for geometry that moves from static to dynamic. Our conjectures around the inclusion of the clinical interview (Adaptation 2) were that teachers would gain better understanding of students' reasoning and mathematical processes. The inclusion of exploratory lessons (Adaptation 3) fulfilled a number of other anticipated outcomes including participants gaining an appreciation of the effects of different pedagogical approaches and the nuances of lesson design. And, importantly, we anticipated that the design and teaching of exploratory lessons would support the PLT's to gain insight into children's mathematical development. Finally, through the creation of resources (Adaptation 4), we conjectured that these activities would support teacher agency and ownership of the process as well provide the PLTs with authentic opportunities to review and reflect on the overall process and their overall learning.

3 A case study of one Professional Learning Team's experience

For the remainder of this paper we present a case study of one PLT. We begin by describing the membership of the team, the school context and student population. Next, we describe the structure, timeframe, and content of the PD process. We describe the research methods and approach to data collection. Finally, in the Results section, we present the specific content of each adaptation along with a qualitative analysis of how the adaptations supported teacher learning.

3.1 Methods and procedures

3.1.1 Participants: the Professional Learning Team

Participants included four junior and senior kindergarten teachers, one first grade teacher, the school principal, a school board numeracy facilitator, and a student achievement officer for the Ontario Ministry of Education.[2] All participants were Caucasian females with a Bachelor of Education degree. The teachers' classroom experience ranged from 3 to 16 years. At the outset of our PD, all teachers expressed a lack of interest and knowledge, as well as skills, in mathematics. Furthermore, several members expressed a deep anxiety about mathematics teaching in general. All teacher members self-described their teaching practice in mathematics as rote and procedural, following the instructional sequences laid out in the district approved mathematics textbook. The teacher team was candid in their resistance to the Ontario Ministry of Education's push towards 'inquiry-based' teaching, claiming these approaches inappropriate for their school population serving families of low socioeconomic status (SES) with a high proportion of English Language Learners (ELLs). Interestingly, this belief in teaching through a rote and procedural approach has previously been described as a prevailing view of teachers of low SES children (Lee and Ginsburg 2009).

3.1.2 School context

The school is located in a large urban Canadian city. The student population consists almost entirely (>90 %) of new immigrants and refugees, the majority of whom have emigrated from Syria and Iraq. The majority of these immigrant students had experienced no formal schooling prior to attending the school. Approximately 65 % of the school's students speak English as a second language. In terms of academic performance, the school consistently places at the bottom of its district and well below the Provincial average on standardized tests in literacy and mathematics.

[2] Henceforth, the term PLT will signify reference to the entire team, including the principal, numeracy facilitator and student achievement officer. We will use the term 'teacher team' to exclusively refer to the classroom teachers.

Table 1 An overview of the PD sessions with the PLT

Session	Record of events
Day 1: full-day	Participated in focus group interview on geometry and spatial reasoning
	Engaged in geometry and spatial reasoning challenges
	Studied video examples of clinical interviews and students engaging in spatial activities
	Designed and conducted clinical interviews and planned exploratory lessons
Day 2: full-day	Viewed videos of clinical interviews and exploratory lessons
	Designed/adapted exploratory lessons and field-tested lessons with JK-Grade 1 students
	Debriefed lessons and modified for future iterations
Day 3: half-day	Taught pattern block symmetry lesson in variety of JK-Grade 1 classrooms
	Debrief lessons and planned lesson extensions
Day 4: full-day	Viewed and analyzed videos of lessons carried out since last meeting
	Tested new lessons with students
	Debriefed lessons and planned future iterations
Day 5: half-day	Reflected on work to date and began planning public research lesson
Day 6: full-day	Conducted exploratory lessons on symmetry in multiple classrooms
	Debriefed/reviewed exploratory lessons and prepared a comprehensive lesson plan
	Prepared presentation and structure for public research lesson
Day 7: full-day	Presented background knowledge underlying the public lesson to audience members
	Presented public lesson
	Debriefed lesson, first with audience members and then without
	Participated in focus group interview on geometry and spatial reasoning
	Gathered resources and reflected on relevant material to include in iBook for others

JK Junior Kindergarten

3.1.3 General overview of the professional development meetings

The research study took place over a 5-month period, spanning January to June. In total, the PLT worked together over 7 days of PD, with 5 full-days and 2 half-days of paid teacher release. All sessions were held in the school library.

Table 1 provides a summary of events for each of the seven meetings. In general, the PD followed the sequence outlined above in Fig. 1. While our meetings varied in content and purpose (see Table 1), meetings shared the common features of: open discussions around mathematics (as both teachers and learners); planning and designing lessons and activities to elicit and promote students' mathematical thinking; and carrying out the lessons and activities with individual or small groups of children and discussing our observations.

3.1.4 Data collection

A mixed-methods approach was used to collect data throughout the PD. Although data were collected from both students and members of the PLT, the present paper reports only on the qualitative data pertinent to the learning of the PLT. Hereafter, evidence of student learning is only considered in the context of the PLT's own learning.

To collect data on the PLT, a wide variety of methods were employed, including video recordings, audio recordings, field notes, and focus group interviews. A handheld video camera was used to capture all meeting events that related to the teaching and learning of geometry and spatial reasoning. The only events not recorded were those deemed not directly relevant to the PD (e.g., scheduling future meetings and off-topic conversations). Audio recordings were used to capture the content of each PD meeting in its entirety. All audio was captured through the use of audio recording programs available on laptop computers. Field notes were completed by two trained research assistants. The field notes included descriptions of the meeting's events as they unfolded in real time as well as an overall summary of each meeting. Focus group interviews were used on the first and last day of PD to gain insight into the PLT's understanding, concepts, and questions and concerns related to the topic of geometry and spatial reasoning. On the final day of PD, PLT members were asked to reflect on their experience with the PD process. Together, the above qualitative data sources formed the basis for the description of the lesson study adaptations reported hereafter.

3.1.5 Data analysis

Video and audio recordings, including focus group interviews, were first transcribed and later coded according to the four lesson study adaptation categories. Two research assistants reviewed the transcripts and then subsequently highlighted and categorized all relevant information into the four adaptation areas. The unit of analysis was the PLT. In cases where the data source was not exclusive to a single category, data were repeated and cross-listed across categories. Coding the transcripts according to the four categories (i.e., lesson study adaptations) was necessary as members of the PD did not necessarily refer their comments to one of the four categories. For the present study, all but one PLT quotation was recorded during the focus group interview conducted during the last day of PD.

The analysis of the video and transcripts during the fourth adaption—creation of resources—were further analyzed for events and evidence that supported our original conjectures of: (1) changed perceptions of geometry and spatial reasoning, (2) changed perceptions of children's thinking and mathematical development, and (3) changed pedagogical approaches to teaching and learning. The fourth adaptation was selected as the source of these major themes because it was during this phase of the PD that teachers shared, discussed, and reflected on the PD process as a whole.

4 Results and discussion: description and qualitative results of individual adaptations

In the following section, a brief description of each adaption is offered, followed by a summary of the main results. The results are qualitative in nature and describe key teacher development and learning outcomes with respect to each adaptation.

4.1 Adaptation #1: first-hand mathematical experience: teachers 'doing' the math

To begin our PD, the PLT engaged in a series of geometry and spatial reasoning activities centered around transformational geometry, spatial visualization, and the composition and decomposition of 2D shapes and 3D figures. The activities were intended to be suitable for the PLT members, yet also adaptable for the young students in their own classrooms. Below is a summary of each activity tried by the members of the PLT.

1. 3D Geometry—*Cube challenge:* Members of the teacher team were provided with multi-link cubes and asked to build as many unique three-dimensional (3D) figures as possible using sets of five cubes. The task required members to identify 3D equivalence through comparing figures of different orientations with the added challenge of understanding and recognizing mirror images.
2. 2D Geometry—*Tangram puzzles:* Members engaged in Tangram challenges, providing the PLT with experience with transformations and composition/decomposition of 2D shapes.
3. 2D Geometry—*Pattern blocks compositions:* Members were provided with outlines of two-dimensional (2D) images that could be filled with various combinations of pattern blocks. Members were asked a series of questions that required visualization skills. E.g., "Just by looking, what is the greatest, what is the least, number of pattern blocks required to fill the space?" This activity provided members with first hand experience *visualizing* the composition and decomposition of geometric images.

Results The 3D geometry activity was challenging for the majority of the group members who shared that they had not previously been exposed to challenges of this kind. Members commented on the role of visualization and mental manipulations as a central aspect of completing the activity. In particular, the group found it difficult to operationalize "mirror images". The recognition of the role of visualization and mental transformations in the 2D geometry activities was also apparent to the group. This was particularly true in the challenge of composing a square with all seven Tangram pieces.

The PLT recognized that their view of geometry teaching and learning did not include a focus on the more dynamic aspects of geometry, such as visualization and mental transformations. As one of the kindergarten teachers remarked during the final focus group session: "We found that in engaging in the activities, that's what opened our eyes to, 'oh, there's a lot more going on here than just the identification and classification of shapes". In the words of another kindergarten teacher, "…it (working with the geometry activities) taught us so much about geometry and spatial sense. I don't think of it anymore as 2D shapes, 3D shapes. It goes above and beyond that". In both these comments, we can see that the teachers valued engaging with the mathematics tasks as a way of broadening their understanding and definition of geometry. During our final focus group together, members of the PLT spoke candidly about the dynamic nature of geometry, stating the need to capitalize on and further develop children's abilities to "visualize", "mentally rotate" and "transform" geometric objects. In our view, this shift towards a more dynamic and spatial view of geometry was at least in part due to the PLT's first-hand experience with the various geometry challenges.

Not only did engaging with the novel tasks result in the group's reconsideration of geometry, but engaging with the tasks also prompted questions about how their young students might respond to similar challenges. Indeed, as further sections of this paper reveal, these activities—and other ones implemented throughout the PD—provided a foundation for the design and teaching of classroom

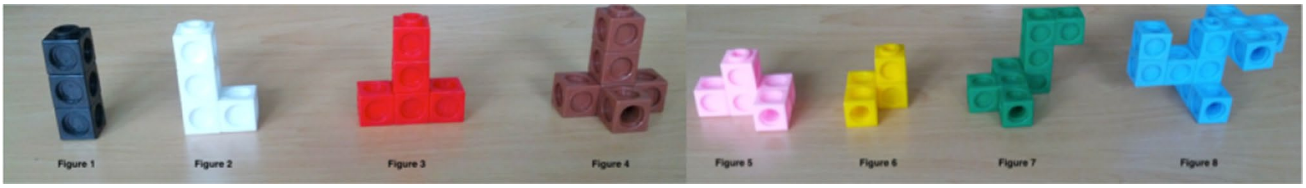

Fig. 2 The PLT's 3D figure composition task. Students were asked to build a replica of each of the above figures

lessons and activities. As the quote below reveals, the PLT were excited to share the activities with their students and their own children:

> "We were having so much fun doing the activities. And it's the exact same thing that happened to our students…I've taken the activities we've done here, a lot of us have, and taken them home to our own children and they're sitting on the coffee tables. It's like, 'no, you need to do this tonight, ….' because it's really changed the way we think about math, in general, and in particular, geometry and spatial sense".

The above quotes, representative of all team members, illustrate how the specific math challenges, involving visualization and mental transformations, made a strong impact on the team. Through participation in these activities, the PLT came to see the importance and usefulness of engaging in the mathematical tasks before having their students do so. According to the school principal, this was an important aspect of the PD and one that differentiated it from other types of PD:

> "I think a key piece to this PD is that the teachers are engaging in the math, and this is where a tremendous amount of learning has occurred… So, rather than have staff attend PD sessions where we're told what you should be doing, we first and foremost got together to look at mathematics. And this is really critical. As teachers we actually engaged in the mathematics, in the materials, in the manipulatives. We're actually working and doing the math before we even consider bringing the math to our students".

4.2 Adaptation #2: teacher-researcher co-designed task-based clinical interviews

Having gained firsthand experience with various geometry and spatial reasoning activities, the team discussed how their own students might perform on tasks requiring similar skills, including visualization and the composition and decomposition of 2D shapes and 3D figures. As a means to carefully observe their students' thinking, we introduced the practice of clinical interviews. After watching videos that modeled the clinical interview purpose and process, the PLT developed two different sets of interview tasks.

The first set of tasks probed students' abilities to visualize and was based on Clements and Sarama's (2009) developmental trajectory for composing and decomposing 2D shapes with pattern blocks. The second set of clinical interview tasks involved students in recreating 3D figures of progressing difficulty (see Fig. 2). The PLT was interested to see, firstly, whether their students were capable of re-creating the figures, and secondly, to observe students' various strategies. Since none of the teachers had ever posed challenges of this nature to their children, the team was uncertain about how their students might perform but in general had very low expectations.

Each teacher member of the PLT carried out the clinical interviews with six of their own students: two "low"-, two "middle"-, and two "high-performing" math students, as deemed by each classroom teacher. All interviews were video recorded for subsequent analysis and team viewing purposes. Teachers were asked to select and share noteworthy examples of their students' performance.

Results The experience of conducting the clinical interviews appeared to lead to significant learning for the team. As the teachers became more experienced as interviewers, we noted that they became more adept at asking probing questions and were better able to provide scaffolds and supports to maximize their students' performance. Surprising for the team, were the number of students, even at kindergarten level, who could complete even the most challenging items of the tasks. The classroom teachers were especially surprised at how well their "low" achieving students performed on these spatial tasks, performing in ways indistinguishable from their "mid" and "high" math attaining peers. Moreover, the clinical interview offered a new method for assessing their students. As the quote below from a kindergarten teacher reveals, the PLT came to see the benefits of conducting clinical interviews; both to gain an understanding of the current state of their students' learning as well as to plan for future instruction.

> "When it comes to math we don't ever sit down with our students one-on-one. But when it comes to reading assessment we do it all the time. We pull students aside and assess what they know and what they've learned and really look at specific areas of difficulty, or strengths. The clinical interviews can help us bet-

ter understand what kids know and don't know in math, which helps. It helps us know what to teach and where to go".

Other group members echoed this statement, adding that the clinical interview process is especially useful for gaining a better understanding, and perhaps a more accurate appraisal, of their typically 'low' achieving students, as well as their quieter students. Finally, the clinical interviews served the purpose of increasing the group's understanding of how children compose and decompose 2D/3D shapes, and thereby, provided empirical grounds and motivation to design and implement these kinds of activities in their classrooms.

4.3 Adaptation #3: exploratory lessons and activities

Between meetings 3 and 6, the PLT designed a set of lessons that involved the exploration of reflection symmetry. The interest in this topic had arisen from the teachers' observations of their students' work with pattern blocks during the clinical interviews. Furthermore, the kindergarten teachers shared examples of children's spontaneous use of symmetry in their block play. Wanting to build on their students' seemingly intuitive grasp of symmetry, the team set out to accomplish what eventually became an exploration of a trajectory of how students approach and learn about symmetry.

To begin their exploration, the PLT designed and carried out a series of activities that required children to complete the other half of symmetrical images, first with magnetic pattern blocks and then with magnetic square units on a coordinate grid. The team designed a progression of activities that started with vertical bilateral symmetry, followed by horizontal bilateral symmetry, and finally, oblique bilateral symmetry. Following the team's exploration with pattern block symmetry, the team began experimenting with grid symmetry and followed the vertical to horizontal progression (see Fig. 3). Indeed, while the pattern block activities extended far beyond curriculum expectations, the lessons on the grid were more challenging, requiring the use of co-ordinates and positional language to describe symmetry. One way of further developing students' positional language was the teachers' decision to provide visual aids such as depictions of hands to represent the left and right side of the line of symmetry.

Not only did the exploratory lessons provide a new context for gaining significant content knowledge for teaching symmetry, but also the design and testing of these lessons provided a forum for discussing pedagogy.

Results A central outcome of engaging in exploratory lessons and activities was an increased focus on observing children's reasoning and as a result, new insights into pedagogy. A key change in teachers' capacity to observe was the gradual decrease of relying solely on language-based assessment of children's understanding. Over time, the PLT came to recognize evidence of learning through non-verbal displays of knowledge. For example, members began to observe and see the importance of gestures as a window into children's mathematical understanding.

Research has shown that the ability to attend to student reasoning is critical but not easily achieved (Jacobs et al. 2010). The series of exploratory lessons created by the team were designed to investigate incremental growth in students' abilities to reason about symmetry. In the PLT's role as teacher researchers, there were many authentic opportunities for close observations of both student reasoning and engagement.

Indeed, because of the limited developmental information in mathematics education literature in relation to symmetry in early years (Sinclair 2008), the PTL were in effect developing a trajectory of how children progress in their understanding of symmetry. Importantly, all members of the team noted how much the children enjoyed working with symmetry, in both lessons and in their everyday play.

Through engaging in exploratory lessons the PLT grappled with and came to reconsider pedagogical approaches for the teaching and learning of young children. The iterative practice of conducting and observing a sequence of similar lessons on symmetry provided opportunities to carefully observe students' learning and subtle effects of different pedagogical moves. In the process of analyzing and revising the exploratory lessons, the PLT engaged in questions dealing with teaching approaches to a topic that was new to all of the children. One view amongst the group was that it is necessary to provide children, especially

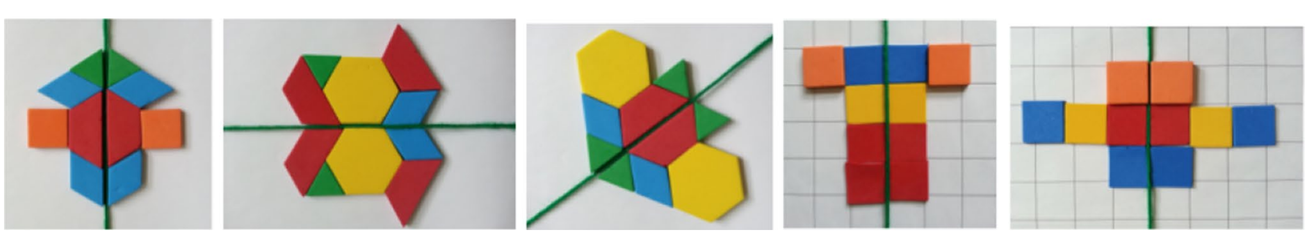

Fig. 3 Symmetrical constructions built by young children

ELLs, with an initial definition of symmetry in order to ensure successful learning. Other members of the team felt that concepts and rules of symmetry should be presented in a way that allowed children to infer the meaning of symmetry. These discussions led to experimenting with the two different approaches. Through this experimentation, the team reached consensus that children assimilate content-specific vocabulary through meaningful and playful learning activities.

4.4 Adaptation #4: creation of resources

The fourth Lesson Study adaptation involves the creation of resources for other early years teachers. The PLT not only designed a presentation and Lesson Study package, but also created their own iBook for others to access and download online (to download a copy visit the link at the bottom of this page[3]). By presenting their work as an iBook, the team could effectively share their learning with others through a combination of print, pictures, and videos.

Results Through creation of resources it was clear that the PLT had gained new perspectives, as well as content knowledge, of geometry and spatial reasoning. For example, the team wished to share with others their new understanding of the limitations of the geometry curriculum and to relate their findings of the impressive competencies and engagement young children bring to their learning of geometry. The team also wanted to include and were excited to share novel geometry activities that they had designed as part of the exploratory lessons. As the content of the iBook reveals, these activities focus on spatial reasoning, visualization, transformational geometry, and congruence. Additionally, the team wished to provide a progression of symmetry activities that offer other educators a more comprehensive and rigorous trajectory for the teaching and learning of early symmetry.

Furthermore, as we conjectured, the creation of the resources supported the PLT's sense of confidence and expertise for teaching and learning of geometry and spatial reasoning.

A key example of this was the teachers' initial willingness and eventual eagerness to publicly share their knowledge with other educators through conference presentations. As noted earlier, this eagerness to present is especially noteworthy considering the team members' initial hesitancy towards the teaching and learning of mathematics in the early years. The team entered the project with low confidence, low interest in mathematics, and low expectations for their students' success in mathematics.

[3] http://www.oise.utoronto.ca/robertson/Inquiry-based_Mathematics/Math_For_Young_Children/St._Andrew_Catholic_School.html.

5 Overall themes of teacher learning that emerged in final meetings

The act of creating resources (Adaptation 4) not only led to the specific results described above, but also provided a context for the PLT to reflect on the PD process as a whole. In our review of the transcripts and video footage in these final sessions, three main themes emerged in the PLT's discussions: (i) children's thinking, (ii) pedagogy, and (iii) the geometry and spatial sense curriculum. In the next, and final section, we elaborate on each theme in turn.

5.1 Changed perceptions of children's capacities

The PLT discussed how the PD process influenced their view of children's capacity to engage in and perform mathematics. Members were surprised at how much more capable children were than they had expected at engaging in seemingly sophisticated mathematics.

> "As someone coming from upper grades, these 4- and 5-year-olds are more capable than I think we've given them credit for in the past. I mean, they're able to take the manipulatives, the concepts, the ideas that we're sharing with them, and just run with them. And they've been able to do a lot of things we didn't expect. They're doing some things we couldn't do ourselves at first".

Adding to this idea, another team member admitted that it was not only surprising to see young children exceeding their expectations but also humbling: "*because we engaged in all of the activities, we thought we knew what to expect from our kids. But when I was trying to come up with all 12 of the pentominoes* (2D shapes composed of 5 square tiles), *I could only come up with 9. Four-year-olds, they can do all 12*!"

In a similar vein, the team also made a distinction in their expectations for typically 'low' achieving students, sharing their observation that several children typically low in numeracy and language demonstrated strengths across various spatial thinking activities and lessons. One final expectation that was overturned as a result of our work together, was the belief that young children are incapable of sitting as a large group and engaging in mathematics for long durations of time (>1/2 an hour). It became clear that even young children (kindergarteners) find enjoyment in engaging in appropriately challenging tasks.

5.2 Rethinking pedagogy

The team referred to their growing understanding of inquiry-based teaching and learning. During the focus group interviews, members admitted that they were skeptical of the benefits for their students of an open-ended, inquiry-based approach to mathematics learning. With respect to pedagogy, the team repeatedly stated the importance of 'learning to observe' children's thinking across a variety of contexts and to look for signs of understanding not only with words but communicated through the body and use of gestures. Both the clinical interviews and the cyclical and experimental nature of the exploratory lessons were referred to as key experiences in learning to observe. The following quote from the school principal suggests the importance of careful observation and its informative role in determining subsequent pedagogical moves or action:

> "When we're in the classroom…what language do we hear? What body language do we see? What do we notice about their learning? And we bring that all back to the table. We observe, we reflect, we kind of re-think and re-tinker some of the activities to think about how can we make it better, how can we make it more accessible to our learners? After each lesson, we reconvened and looked at what the kids did, what we heard, saw and how we could improve this lesson for the kids?"

However, as a result of the PD, members came to see the importance of flexibility in their teaching, the strengths of good questioning, and being responsive to children's mathematical learning behaviors.

One of the key reference points for the team, cited earlier, was the finding that children were more engaged and demonstrated a deeper understanding of symmetry when the term 'symmetry' was not explicitly explained to the children, but was inferred through means of a game-like activity in which the term symmetry was used informally as part of student and teacher interactions.

5.3 New perspectives on geometry

One final theme that emerged as a result of the PD was a new perspective on the early years geometry and spatial sense curriculum. Teachers discussed how their previous experience teaching geometry and spatial sense was limited to teaching the names of shapes, counting the sides and faces, and dealing primarily with static 2D and 3D shapes. All members were adamant in their belief that children are capable of exceeding the curriculum expectations if provided with the proper learning opportunities:

> "Of course we were cognizant of the early learning document curriculum expectations during our learning, (but) we didn't allow ourselves to design activities just to meet those criteria. Instead, as a group, we considered what are the big ideas in geometry and spatial sense, and what are activities that will address those big ideas? How can we present them to the students and let's see what they do with them. Not only did they achieve what is written in the curriculum documents, but many of the expectations that we see later on in the primary grades, and actually even into some of the junior grades, our kids were demonstrating expectations".

Members of the PLT admitted that they would never approach geometry in the same way, and instead, saw the need to also introduce young children to ideas and opportunities to engage in tasks that relate to a more dynamic and transformational approach to geometry and spatial sense.

These themes that emerged during the team's discussions and reflections of the PD process closely align with our overarching initial research objectives: to support early years teachers to gain content knowledge of geometry and to broaden teachers' understandings of their students' capabilities and interests in geometry and spatial reasoning.

6 Concluding words

The challenge of supporting early years teachers to provide rigorous, challenging mathematics instruction in early years classrooms has been extensively reported in the early years and mathematics education literature. In this paper we report on our PD research project designed to address these issues, particularly in the teaching and learning of geometry and spatial reasoning in the early years. Our PD model, based on the Japanese Lesson Study cycle, included four adaptations; teachers engaging in mathematics, teachers conducting co-designed task-based clinical interviews, teachers and researchers designing and carrying out exploratory lessons, and the creation of resources for other educators. These adaptations, based on and closely aligned with proven and well-established features of PD, were included to address the specific challenges of supporting early years educators in mathematics instruction. To illustrate the implementation and effects of our adaptations we presented a case study of one PLT. At the outset of the study, the participants exhibited many of the attitudes and concerns about mathematics teaching typically reported in the literature, including a lack of confidence, interest and specific mathematics knowledge, as well as a general underestimation of young students' capabilities. Indeed, our results revealed that our approach to PD—with its adaptations—was successful in supporting members to gain a deep content knowledge and broadened conceptualization of geometry and spatial reasoning. In addition, team members expressed a newfound interest and commitment to teaching and learning of geometry. Lastly, the PLT came to recognize and acknowledge their young students' sustained interests and capacity to engage in sophisticated geometry and spatial reasoning curricula.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bruce, C. D., & Ladky, M. S. (2011). What's going on backstage? Revealing the work of Lesson Study with Mathematics Teachers. In *Lesson Study Research and Practice in Mathematics Education* (pp. 243–249). Netherlands: Springer.

Bruce, C., Moss, J., & Ross, J. (2012). *Survey of JK to Grade 2 teachers in Ontario, Canada: Report to the Literacy and Numeracy Secretariat of the Ministry of Education*. Toronto: Ontario Ministry of Education.

Casey, B. M., Andrews, N., Schincler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26*(3), 269–309.

Clarke, D., Clarke, B., & Roche, A. (2011). Building teachers' expertise in understanding, assessing and developing children's mathematical thinking: the power of task-based, one-to-one assessment interviews. *ZDM–The International Journal on Mathematics Education, 43*(6–7), 901–913.

Clements, D. H. (2004). Geometric and spatial thinking in early childhood education. In D. Clements, J. Sarama, & M. A. DiBaise (Eds.), *Engaging young children in mathematics: results of the conference on standards for pre-school and kindergarten mathematics education* (pp. 83–90). New Jersey: Erlbaum.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: summative research on the Building Blocks project. *Journal for Research in Mathematics Education, 38*(2), 136–163.

Clements, D. H., & Sarama, J. A. (2009). *Learning and teaching early math: the learning trajectories approach*. New York: Routledge.

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of Mathematics Teacher Education, 14*(2), 133–148.

Cobb, P., Jackson, K., & Munoz, C. (2015). Design research: a critical analysis. In L. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (3rd Ed.). Manuscript submitted for publication. New York: Taylor & Francis (in press).

Copley, J. V. (2004). The early childhood collaborative: a professional development model to communicate and implement the standards. In D. H. Clements & J. Sarama (Eds.), *Engaging young children in mathematics: standards for early childhood* (pp. 404–414). New Jersey: Erlbaum.

Cross, C. T., Woods, T. A., & Schweingruber, H. (Eds). (2009). *Mathematics learning in early childhood: paths toward excellence and equity*. *Committee on Early Childhood Mathematics.* National Research Council and National Academy of Sciences.

Darling-Hammond, L., Chung Wei, R., Andree, A., & Richardson, N. (2009). *Status Report on Teacher Development in the United States and Abroad features of good professional development.* United States of America: National Staff Development Council and The School Redesign Network at Stanford University.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., & Japel, C. (2007). School readiness and later achievement. *Developmental Psychology*, *43*(6), 1428–1446. doi:10.1037/0012-1649.43.6.1428.

Freudenthal, H. (1981). Major problems of mathematics education. *Educational Studies in Mathematics Education., 12*(2), 133–150.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915–945.

Ginsburg, H. P. (1997). *Entering the child's mind: the clinical interview in psychological research and practice*. New York: Cambridge University Press.

Ginsburg, H. P., Cannon, J., Eisenband, J., & Pappas, S. (2006). Mathematical thinking and learning. In K. McCartney & D. Phillips (Eds.), *Blackwell handbook of early childhood development* (pp. 208–229). Massachusetts: Blackwell Publishing.

Ginsburg, H. P., Lee, J. S., & Boyd, J. S. (2008). Mathematics education for young children: what it is and how to promote it. *Social Policy Report of the Society for Research in Child Development, 22*(1), 1–23.

Hachey, A. C. (2013). The early childhood mathematics education revolution. *Early Education and Development, 24*(4), 419–430.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). The effects of teacher's mathematical knowledge for teaching on student achievement. *American Educational Research, 42*(2), 371–406.

Huang, R., Li, Y., Zhang, J., & Li, X. (2011). Improving teachers' expertise in mathematics instruction through exemplary lesson development. *ZDM–The International Journal on Mathematics Education, 43*, 805–817.

Jacobs, V. R., Lamb, L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal of Research in Mathematics Education, 41*(2), 169–202.

Jones, K. (2000). Critical issues in the design of the geometry curriculum. In Bill Barton (Ed.), *Readings in Mathematics Education* (pp. 75–90). Auckland: University of Auckland.

Lee, J. (2010). Exploring kindergarten teachers' pedagogical content knowledge of mathematics. *International Journal of Early Childhood, 42*(1), 27–41.

Lee, J. S., & Ginsburg, H. P. (2009). Early childhood teachers' misconceptions about mathematics education for young children in the United States. *Australian Journal of Early Childhood, 33*(4), 37–45.

Levenson, E., Tirosh, D., & Tsamir, P. (2011). *Preschool Geometry: Theory, Research and Practical Perspectives*. Boston: Sense Publishers.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of Lesson Study. *Educational Researcher, 35*(3), 3–14.

MacDonald, A., Davies, N., Dockett, S., & Perry, B. (2012). Early childhood mathematics education. In B. Perry (Ed.), *Research in mathematics education in Australasia 2008–2011* (pp. 169–192). Boston: Sense Publishers.

Mix, K. S., & Cheng, Y. L. (2012). Space and math: The developmental and educational implications. In J. B. Benson (Ed.), *Advances in child development and behavior* (pp. 179–243). New York: Elsevier.

Moss, J., Messina, R., Morley, E., & Tepylo, D. (2012). Sustaining professional collaborations over 6 years: Using Japanese Lesson Study to improve the teaching and learning of mathematics. In J. Bay-Williams & W. R. Speer (Eds.), *Professional collaborations in mathematics teaching and learning: Seeking success for all* (pp. 297–309). Reston: The National Council of Teachers of Mathematics Inc.

National Association for the Education of Young Children. (2010). *Early childhood mathematics: Promoting good beginnings.* Washington, DC: NAEYC. Retrieved 3 March 2014. http://www.naeyc.org/files/naeyc/file/positions/psmath.pdf.

National Council of Teachers of Mathematics. (2000). Chapter 4: Standards for pre-K-2. In *Principles and Standard for School Mathematics.* Reston: NCTM.

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics.* Reston: Author.

National Research Council. (2006). *Learning to think spatially: GIS as a support system in the K-12 curriculum*. Washington, DC: National Academic Press.

Newcombe, N., Uttal, D., & Sauter, M. (2013). Spatial development. *Oxford handbook of developmental psychology, 1*, 564–590.

Sinclair, N. (2008). *The history of the geometry curriculum in the United States*. IAP—Information Age Publishing Inc.

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). The malleability of spatial skills: a meta-analysis of training studies. *Psychological Bulletin, 139*(2), 352–402. doi:10.1037/a002844.

Van den Heuvel-Panhuizen, M. (2008). *Children learn mathematics: a learning-teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school*. Utrecht: Freudenthal Institute.

Van den Heuvel-Panhuizen, M., & Buijs, K. (2005). *Young children learn measurement and geometry: a learning-teaching trajectory with intermediate attainment targets for the lower grades in primary school*. Utrecht: Freudenthal Institute.

Verdine, B. N., Golinkoff, R., Hirsh-Pasek, K., Newcombe, N., Filipowocz, A. T., & Chang, A. (2013). Deconstructing builidng blocks: Preschoolers' spatial assembly performance relates to early mathematics skills. *Child Development*. doi:10.1111/cdev.12165 (Advance online publication).

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., & Newcombe, N. S. (2014). Finding the missing piece: Blocks, puzzles, and shapes fuel school readiness. *Trends in Neuroscience and Education*. doi:10.1016/j.tine.2014.02.005 (Advance online publication).

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: aligning over fifty years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology, 101*, 817–835.

White, A. L., & Lim, C. S. (2008). Lesson study in Asia Pacific classrooms: Local responses to a global movement. *ZDM–The International Journal on Mathematics Education, 40*(6), 915–925.