BUILDING PER

Connecting Linear Measurement

JOAN MOSS AND BEVERLY CASWELL

THERE WAS AN AIR OF EXCITEMENT and anticipation in the grade 5/6 class as the students consulted with one another and put the final touches on their percent measurement dolls. The doll-making unit, a favorite with the students, was a culminating activity in an ongoing research project for learning rational number and proportion. The students, who attend a laboratory school associated with the Ontario Institute for Studies in Education, and their teacher, Beverly Caswell, had just spent the last five mathematics classes working on the measurement, design, and building of these dolls. They had been invited to present their creations to a group of preservice teachers and to explain the mathematics that had been involved.

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Alice was the first to speak. "Hi everybody. This is Paige McMillan," she said, holding up her colorful doll. "She is 60 cm tall and she is proportional. She is a scale model of what the average of our grade 5/6 class's proportions look like. She and the other percent dolls in our class are made of the average measurements of body percents of our entire class. How we did it was we measured our heights and then we measured lots of body parts and found out what percent, say, the height of our head (from the chin to the top of our foreheads) was compared to our height. So it's about an eighth of my total body, 'cause I am 140 cm tall and my head height is 17.5 cm so it's 12 1/2%. So for our doll Paige, we had to measure 7 1/2 cm for her head height 'cause that is 1/8 of 60 cm. Then we averaged the percents of our entire class and we got to make our dolls."

When Nick presented his Pinocchio doll, he explained that "the whole doll is proportional, even

CENT DOLLS:

to Learning Ratio and Proportion

the nose," but he acknowledged that there was a problem with the fishing pole. "I realize now that it really isn't proportional. See, it is shorter than the doll but my own fishing pole at home is about 150% of my height, so this one's way off."

The preservice teachers were eager to learn more. How long had it taken the students to make these dolls? What was the sequence of activities that led to their completion? What kinds of activities had preceded the doll building? As they could hear from the students' descriptions, although the doll-making project had its basis in linear measurement, the mathematics that the students had used also involved performing rational number calculations, translating among fractions and percents, and working out mathematical averages. How had these topics been approached, and why did the children use percents to describe the mathematical relationships instead of fractions as was usually done with body proportion activities of this kind? (e.g., Burns 2000).

In this article, we describe the percent-doll unit through details of the lessons and accounts of the students' learning. To set the context for these lessons, we begin with a brief discussion of our ongoing research project and how linear measurement served as a starting point for our work on rational-number learning with the students in our research classrooms.

Linear Measurement to Support Learning Rational Numbers

AS THE NCTM (2000) POINTS OUT, THE LEARNING of measurement not only gives students opportunities to apply appropriate tools, techniques, and formulae to determine measurements but the study of measurement facilitates the learning of other topics and allows students to see the connections among mathematical ideas. In our current research program in middle school classrooms, we have been implementing and assessing an approach to teaching the difficult topic of rational numbers (decimals, percents, and fractions) and the aligned topics of ratio and proportion using representations involving linear measurement. We know from substantial research that learning about rational numbers and proportion is often difficult for students: the learner must move from the familiar additive reasoning that underlies whole numbers and counting toward the more complex multiplicative reasoning that underpins rational numbers and proportions (Carpenter, Fennema, and Romberg 1993). Many students appear to confuse these forms of reasoning (Hart 1988).

In our research teaching, we promote the idea of multiplicative relations by designing activities grounded in linear measurement contexts in which students consider ideas of relative amount and fullness. Moreover, in our instruction, rather than introduce rational number concepts through the traditional approach of using decimals and fractions, we use students' everyday knowledge of percents as the starting point. For example, students compare heights in relative terms using language that involves percent ("She is about 75% as tall as he is"), or they compare the fullness of containers relative to the whole ("That cup is approximately 25% full"). Further, they go on to use their informal knowledge of percents and a strategy for halving and doubling numbers to perform calculations using percents: "Your height is 160 cm, then 50% of your height is 80 cm, and 25% of your height is 40 cm." Our research findings have revealed that by using these particular measurement contexts and percents, students naturally consider the ratio relation of one dimension to another and also consider proportional relations (Kalchman, Moss, and Case 2001; Moss and Case 1999, 2002). Thus, they think about the multiplicative and not the additive or absolute relations involved. (For a fuller discussion of additive and absolute thinking, see Lamon 1999 and Moss, in press.)

The measurement exercises and visual displays evoked discussions of proportions

In the sections that follow, we describe the sequence of activities that the students engaged in over two weeks to make their percent measurement dolls. Since the scope of this article does not allow for a description of the full curriculum (see Moss 2000 and 2001 for details of the curriculum), we begin our account of the unit with the description of a series of string measurement activities in which the students took part. We believe that these activities can serve as a good starting place for teachers and students interested in making percent dolls in their own classrooms. We would like to point out that the halving and doubling strategies the students use for estimating and calculating were not taught to them at any point but were methods that the students had learned before beginning the activities discussed here.

String Challenges: Guessing Mystery Objects

THE DOLL-MAKING UNIT STARTED WITH WHAT we called "string challenges." First, we presented students with various lengths of string cut to represent different percentages of the heights of mystery objects in the classroom. The activity was introduced this way:

TEACHER: [*Holding up a small piece of string.*] I have here a length of string that I have measured and cut so that it is 25% of the length of a mystery object in the classroom, and it is within your sight. Any ideas as to what the mystery object might be?

STUDENT: I think that it is the length of desktop or maybe the length of the Bristol board on the wall.

TEACHER: How did you figure that out?

STUDENT: Well, I just imagined moving the string along the desk four times, and I think it works. [*The student then carefully moved the string along the desk and was able to confirm her assertion.*]

Next, the students worked in pairs and challenged their classmates to find the lengths of strings that corresponded to percents of the length of their own mystery objects. Two students, Becky and Scott, for example, chose the classroom pencil bin as their secret object. First they measured the height, 46 cm, then calculated that 25% of the length would be 11 1/2 cm. They cut a piece of string to correspond to that value and challenged their classmates to guess the mystery object. The students enjoyed this activity a great deal, so we expanded it, and the students produced what they called "percent families." They used their secret object measurement as a base, then cut a series of strings that represented the benchmarks of the halving percents. These strings were then labeled and taped onto large sheets. As they had done with the mystery object activity, the students presented their percent families to one another, reviewed the calculations they had done, and challenged their classmates to guess their mystery object. **Figure 1** shows an example of a completed percent families display. The students who produced this item first measured their object, then cut strings that represented the halving benchmarks of 75%, 50%, 37.5%, 25%, 12.5%, and 6.25%.

These measurement exercises and the visual displays that the children created—mounted on the walls around the classroom—naturally evoked discussion of proportions. Students remarked, "Our string lengths are different even though all of our percents are the same."

Body Percents

THESE STRING CHALLENGES, WITH THEIR FOCUS on estimation, measurement, and percents, served as a good introduction to discovering body proportions—an idea adapted from a lesson designed by Marilyn Burns (2000)—and then to designing and building the percent dolls.

The body proportion activities began with the teacher asking this question, "Take a look at my

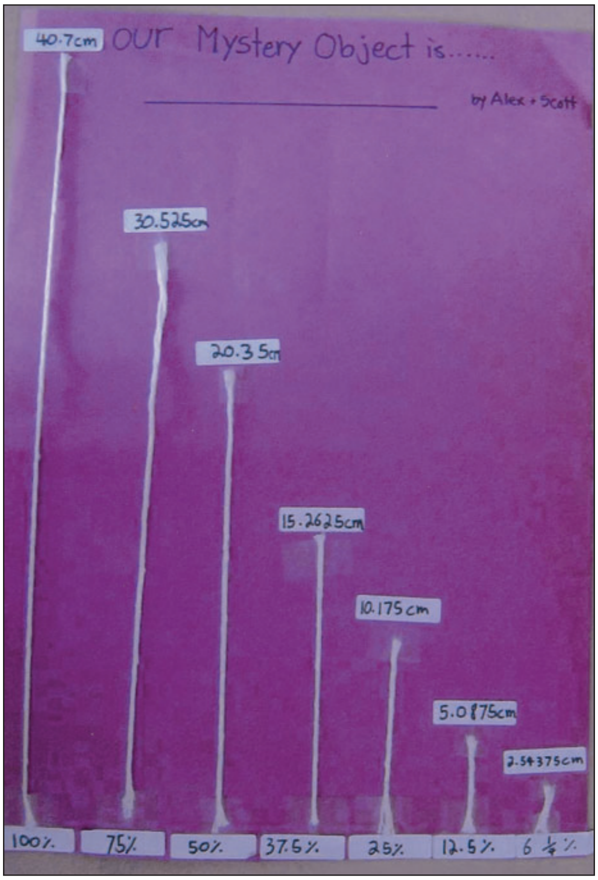

Fig. 1 Mystery object percent families

foot. Can you please tell me what percent of my body height you think my foot is?" (The answers provided by the students ranged from 10% to 20%.) "I have already measured my height—I am approximately 168 cm tall. Let's cut a string that is as long as my foot. Here it is. Let's measure it. It is 23 cm. So now what is that as a percent?" Most of the students relied on a halving strategy to compare the foot with the length as a percent, then they reasoned as follows: "50% of our teacher's height is 84 cm, 25% of her height is 42 cm, and 12 1/2% is 21 cm." This strategy would give them an approximation of the percent of the foot compared with the teacher's height. As one student, Maya, expressed: "I did the calculations for 12 1/2% and that is 21 cm, so 23 cm is pretty close. So, I'd say that your foot is nearly 12.5%." Silas, a grade 6 student, continued with these comments: "But your foot is 23 [cm] so it is a bit more. So if you add 1 more percent that's 1.68 [1% of 168]; you get even closer."

Before the students worked on their own measurements, they practiced by estimating other body proportions of the teacher. For example, after measuring, they discovered that her arm span was nearly 100% of her body height, that the length of her waist to foot was about 60% of her height, and that the circumference of her head was 37.5%.

Body Percents

Name: *Bev* Your Height: *168* cm

What is the measurement of your **arm span?** *168* cm
What percent is that of your height? *100* %

What is the measurement of your **foot?** *24* cm
What percent is that of your height? *14* %

What is the measurement of your **hand?** *17* cm
What percent is that of your height? *10* %

What is the measurement of your **baby finger?** *6* cm
What percent is that of your height? *4* %

What is the measurement of your **head circumference?** *60* cm
What percent is that of your height? *35.7* %

What is the measurement of your **head length?** *21* cm
What percent is that of your height? *12.5* %

What is the measurement of your **shoulder width?** *40* cm
What percent is that of your height? *25* %

What is the measurement of your **waist to floor?** *98* cm
What percent is that of your height? *60* %

Fig. 2 Body percents

Measuring Body Parts and Calculating Percentages

AFTER MEASURING AND ESTIMATING SOME OF the percentages of the teacher's body parts, the students then went on to find the measurements and percentages of their own body parts. With blank data sheets in hand, the students worked in pairs to help each other measure and record the information on the data sheets. To help with their calculations, the teacher suggested that the students create a list similar to the percent families list they had created based on the measurements of their mystery object. This time, however, they would use their own heights as 100% bases. Again, their method was to use a halving operation to find these "benchmark" quantities. As the list in **figure 2** illustrates, students started with 100 and used a series of halving operations to calculate 50%, 25%, and 12.5% of their height. They also used halving based on the 10% quantity to determine what 5% and 2.5% was of their height. However, further calculations

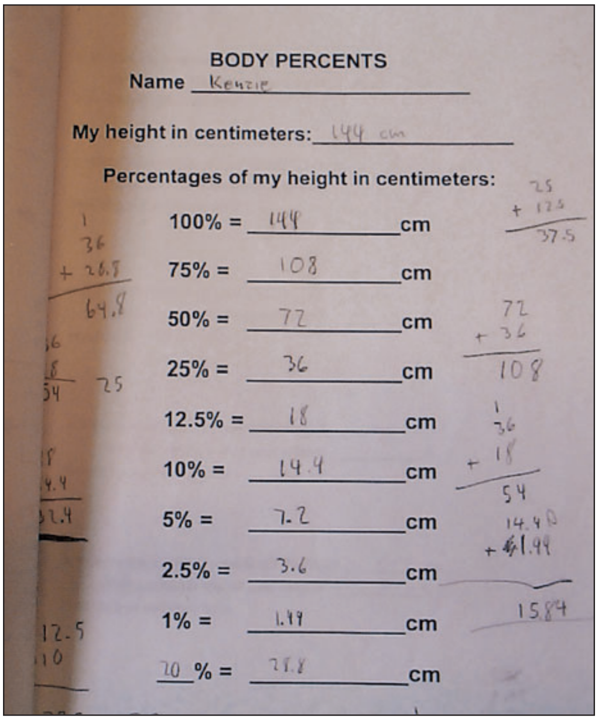

Fig. 3 Benchmark percents based on a student's height

Fig. 4 Class average percentages

were needed to compute the intermediate percents of 75, 37.5, and so on.

How did the students find these percents? A closer look at the addition calculations that Kenzie jotted down in the right margin of his worksheet (see **fig. 3**) shows how he calculated these percentages. To find 75% of his height, Kenzie added 72 cm (50% of his height) to 36 cm (25% of his height) to find the correct answer that 75% of 144 is 108. Similarly, he found the answer to 37.5% of his height by summing 36 cm (25%) and 18 cm (12.5%) to determine that 37.5% of his height is 54 cm.

Students Discuss Proportions

AS THE STUDENTS WORKED IN PAIRS TO MEASURE and calculate their percentages, we noticed that their conversations contained references to proportional reasoning:

PAULO: Hey, my head height is the same percentage as Darnel's even though we're not the same height and we had different answers for the centimeters tall. What did you get for your foot length?

It was interesting for us to observe the level of comfort that the students had in this mixed class when they revealed their body measurements. As many teachers will agree, classroom measurement activities involving body comparisons based on absolute differences can inadvertently make some students feel uncomfortable. For example, the task of comparing heights of the students in the room may disadvantage shorter children, although it is mathematically useful. In our project, students are looking for percents and proportions compared with *their own* height. Thus, any comparisons among classmates are within a similar range, and self-esteem is left intact. For example, "My hand is 10% of my height. What's yours?" (Answers ranged from 7.5% to 12%.)

When all the students had finished calculating their individual proportions, they pooled these percentages and worked out the class average on a large data sheet (see **fig. 4**). Once the average proportions were established, the idea emerged to use these data to make a series of classroom dolls. These dolls would represent scale models of the class average proportions.

Constructing the Dolls

THE CLASS THEN BRAINSTORMED IDEAS ABOUT what materials could be used to build these dolls. The teacher thought of pipe cleaners; students' ideas ranged from wooden dowels and cardboard tubes to swatches of material, yarn, Plasticine, and glitter. Next, the students worked in pairs to negotiate what the dolls would look like. First, they established a general design and listed materials they would like to use. Most important, they selected the height for their dolls. (In the end, the dolls ranged in height from 35 cm to 70 cm.)

With these decisions made, the doll building could begin. However, a new set of calculations was required. Students needed to convert the percents from the class average into a new measurement so that the dolls' body parts would be proportional in relation to their heights. The students were eager to make their dolls as representative of the class average as possible, and they used a variety of strategies to do so. Although some students filled in new data sheets in which they performed these conversions (**see fig. 5**), others worked out the measurements as they built their dolls.

Alice and Paulo, for example, were considering how long to cut the wooden dowel that would serve as the arms and hands of their doll.

ALICE: Now let's see. Our doll is going to be 40 cm tall. I know the [average percentage] shoulder to fingertip is 44%, so hmmm, 10% of 40 cm is 4 cm, so 4 times 4 is 16. That is 40%, so it's a bit more than 16.… It is about 18 cm.

PAULO: Yeah, that makes sense because 50% of 40 is 20 cm. So it must be pretty close. So let's cut that wooden stick 18 cm.

What about making a head for a doll? Joseph and his partner, Krista, had decided to use a balloon for the head of their doll. How would they calculate the distance of the circumference, and how would they expand a balloon so that it had the exact circumference required? The following vignette shows how they reasoned and accomplished this task.

JOSEPH: [*Holding a balloon in his mouth, he begins to blow, then pauses to talk with his partner.*] The class average head circumference is 37 1/2%, so that's the 25% measurement plus the 12 1/2% measurement. It's 3/8. Okay, our doll is 60 cm tall. So for our doll that's 22 and a half cm. So I have to blow up the balloon so that the circumference is 22 1/2. [*Joseph then continued to blow up the balloon while his partner held the measuring tape around the balloon.*]

KRISTA: Okay, Joseph, slow down. You've already blown the balloon to 20 cm. Just a little more. Stop. That's too much, let some out. OK, we're at 22.5 cm. Stop blowing!

Conclusion

THE DOLL-MAKING UNIT WAS A FINAL PROJECT IN a series of twenty-five experimental lessons in which students built their understanding of rational number concepts through working with percents and linear measurement. Our goal for this research project was to present students with a series of activities that would foster a multiplicative understanding of rational number concepts. A further question for this research was whether the mea-

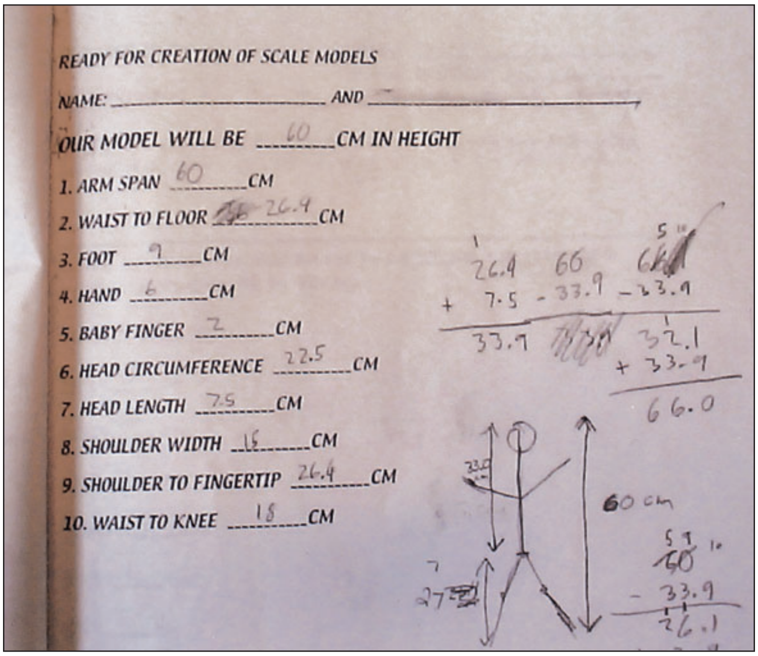

Fig. 5 Scaled percentages for building dolls

surement activities that we designed would help students to develop an understanding of proportional reasoning. In this project, as in our previous experimental studies on rational number understanding, students participated in extensive pretest and posttest interviews in which their knowledge of rational number concepts was assessed. Quantitative results revealed that students not only made significant gains in their ability to work with decimals, fractions, and percents, but that they also made substantial gains in their abilities to perform standard proportional reasoning tasks. Moreover, as can be seen in the protocol of students' reasoning throughout this article, students gained a flexibility in moving among the representations of rational number.

In our project, students look for percents and proportions compared with *their own* height, so comparisons among classmates are largely avoided

As for the doll-making unit, the students seemed to enjoy the process and to value the learning that they did. In their communications with the preservice teachers invited to view the products of their doll making, they showed a pride in their work, both mathematical and artistic. Perhaps one lesson gained from this teaching experience is that working with measurement activities does indeed offer many advantages. Not only do measurement activities ground mathematics in authentic tasks and hands-on experience, but they also promote the understanding of many mathematics topics as well as highlight the interrelations among them.

Throughout the many stages involved in building these dolls, these grade 5/6 students gained practice performing calculations and finding averages in meaningful ways as well as translating among percents and fractions to perform operations. But perhaps most significant was the way that our general approach to teaching rational numbers along with the doll-making unit appeared to help the students formulate their ideas about the meaning of ratio and proportion—topics known for the difficulties they present to students (Lamon 1999).

Sophia, a strong mathematics student, was able to articulate how making the percent dolls had led her to understand proportion. She explained: "A proportion is a part of something. It's basically what one part is of another, basically a fraction. So the head height is 12 1/2%, so it's 1.25 out of 10. And a proportion is something that stays consistent through all different sizes and all different things. It's consistent because it has the same relation, bigger and smaller—it's always related. Our dolls are a lot smaller than the imaginary big one [the class average], but their proportions, every part of the body, is the same. So although they're different sizes, they are the same size equivalent to the body."

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